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On a special case of the Diophantine equation $ax^2 + bx + c = dy^n$

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Abstract. The parity of the positive integer v allows the solvability of the Diophantine equation $13^v - w^2 - w = 1$. When $v \equiv 0 \pmod 2$, it is shown that the former equation has only the trivial solutions $(v,w) = (0,0)$, $(0,-1)$. When $v \equiv 1 \pmod 2$, we use the order $\mathscr{O}_f = \mathbb{Z} + \mathbb{Z}[1+\sqrt{13}]$ of index $f = 2$ of the quadratic field $\mathbb{Q}(\sqrt{13})$ and the recursive sequences to prove that it has only the solutions $(v,w) = (1,-4)$, $(1,3)$.

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1 Introduction

In this paper, we consider the Diophantine equation

$$13^v - w^2 - w = 1 \tag{1.1}$$

in integers v and w. This equation is of the form

$$ax^2+bx+c=dy^n \tag{1.2}$$

where a, b, c and d are fixed integers, $a \neq 0$, $b^2-4ac \neq 0$, $d \neq 0$, which has only a finite number of solutions in integers x and y when $n \geq 3$. This was first shown by the author of [8] (see also [7]). It is well known that there is no general method for determining all integer solutions of (1.2), but many special cases of such Diophantine equations have been studied in the last few years, the discussions involving the following notions:

- the quadratic fields (see for example papers [1,2] and [3]);
- the recursive sequences (see also for example papers [5] and [6]).

We are going to use the above methods to study the solvability of (1.1).
When v is even: $v=2t$, equation (1.1) becomes $13^{2t}-w^2-w=1$ and when v is odd: $v=2t+1$, (1.1) becomes also $13^{2t+1}-w^2-w=1$. Write $z=13^t$. Then we have: when v is even,

$$z^2=(w+1)^2-w; \tag{1.3}$$

when v is odd,

$$13z^2=w^2+w+1. \tag{1.4}$$

Multiplying (1.4) by 4, we obtain $13(2z)^2=4w^2+4w+4$, and so

$$13(2z)^2=(2w+1)^2+3. \tag{1.5}$$

Write

$$x=2w+1, \quad y=2z. \tag{1.6}$$

Then we have

$$x^2-13y^2=-3. \tag{1.7}$$

Equation (1.7) is of the form

$$v^2 - (ka^2 + b^2)w^2 = -k \tag{1.8}$$

treated in paper [4] with $\gcd(a,b) = 1$, $a \geq 1$, $b \geq 0$ and $\delta = ka^2 + b^2$ nonsquare in $\mathbb{Z}$:

$$\delta = ka^2 + b^2 = g^2 d$$

where $g \geq 1$ and d is positive square-free. Write also

$$L = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{\delta})$$

of discriminant D_L: L is a real quadratic field and $\mathcal{O}_L$ the maximal order of L. The discriminant of equation (1.8) is

$$D = 4\delta = 4g^2 d = f^2 D_L.$$

Then, with Theorem 2 of [4] the order of index f associated to (1.8) of discriminant D is $\mathcal{O}f = \mathbb{Z} + \mathbb{Z}f\omega$ with:

$$\omega = \frac{1+\sqrt{d}}{2}, \quad f = 2g \text{ if } d \equiv 1 \pmod 4.$$

or

$$\omega = \sqrt{d}, \quad f = g \text{ if } d \equiv 2,3 \pmod 4.$$

Then, taking $k = 3$, $a = 2$ and $b = 1$ in (1.8), we obtain (1.7). Thus we have $D = 4 \times \sqrt{13}$, $d = \sqrt{13}$, $D_L = \sqrt{13}$; hence $f = 2$ and $\omega = \frac{1+\sqrt{\sqrt{13}}}{2}$.
Therefore, the order of $\mathbb{Q}(\sqrt{\delta})$ associated to equation (1.7) is

$$\mathcal{O}_2 = \mathbb{Z} + \mathbb{Z}[1 + \sqrt{\sqrt{13}}]. \tag{1.9}$$

Hence the solvability of (1.1) will be based on (1.3) and (1.7).

Thus, in Section 2, we give all the solutions of equation (1.3) (Theorem 1). In Section 3, we describe with the aid of arguments of [4] the families of all the solutions of (1.7) (Proposition 2); in the remainder of Section 3, we introduce the recursive sequences connected to these families of solutions (Proposition 3. These sequences allow to prove in Sections 4 and 5 respectively that when v is odd, (1.1) has no solution in Z (Theorems 6 and 8). The paper is ended in Section 6 with the complete set of the solutions of (1.1).

2 Solutions of equation (1.3)

In this section, we begin by giving the integer solutions of (1.3) in integers w and z.

Theorem 1. *Equation* (1.3) *has only the solutions* $(w,z)=(0,1),(0,-1),(-1,1),(-1,-1)$.

Proof. Write (1.3) in the form

$$(w-z+1)(w+z+1)=w.$$

Then we see that $w-z+1$ and $w+z+1$ are divisors of w. Thus, if the sign is chosen so that sgn w= sgn z, then we have

$$|w\pm z+1|\geq ||w\pm z|-1|=||w|+|z|-1|,$$

whence

$$|w|+|z|-1\leq |w\pm z+1|\leq |w|$$

and $|z|\leq 1$. Therefore there are only three values of z:
$z=0$: then, from (1.3) we have $w=\frac{-1\pm\sqrt{-3}}{2}$ which is not an integer;
$z=1$: then, (1.3) $\Rightarrow$w= -1 and $w=0$;
$z=1$: then, (1.3) $\Rightarrow$w= 1 and $w=0$. □

3 Solutions of equation (1.8)

It is well known by [4] that equation (1.8) has two families of solutions. In this section, we use arguments of [4] to prove the following proposition:

Proposition 2. *The solutions in pairs of natural numbers* (x,y) *of* (1.7) *comprise the values of the sequences* (x_w, y_w) $(w \geq 0)$ *by setting:*

$$x_w - y_w\sqrt{\sqrt{13}} = \pm(7 \pm 2\sqrt{\sqrt{13}})(649 \pm 180\sqrt{\sqrt{13}})^w. \tag{3.1}$$

Proof. First, we develop $-f\omega^\sigma = 1+\sqrt{13}(\omega^\sigma$ is the conjugate of ω in $mO_f)$ in continued fraction to find the fundamental unit of mO_f (cf. relation (1.9)):

$$\begin{aligned}
\sqrt{13}-1 &= 2+(\sqrt{13}-3), && u_0 = 2\\
\frac{1}{\sqrt{13}-3} &= 1+\frac{\sqrt{13}-1}{4}, && u_1 = 1\\
\frac{4}{\sqrt{13}-1} &= 1+\frac{\sqrt{13}-2}{3}, && u_2 = 1\\
\frac{3}{\sqrt{13}-2} &= 1+\frac{\sqrt{13}-1}{3}, && u_3 = 1\\
\frac{3}{\sqrt{13}-1} &= 1+\frac{\sqrt{13}-3}{4}, && u_4 = 1.
\end{aligned}$$

Next, we form the following table:

n	0	1	2	3	4
u_n	2	1	1	1	1
p_n	2	3	5	8	13
q_n	1	1	2	3	5
$p_n^2+2p_nq_n-12q_n^2$	-4	3	3	4	-1

The fundamental unit of mO_2 of the quadratic field $\mathbb{Q}(\sqrt{13})$ is:

$$\sqrt{13}+5(1+\sqrt{13}) = 18+5\sqrt{13}.$$

Moreover, since the Legendre symbol $\left(\frac{3}{\sqrt{13}}\right)$ is equal to 1, 3 splits in mO_2 and Proposition 1 of [4] shows that $\mathscr{O}_2$ has two non-associated elements of norm 3:
$7+2\sqrt{13}$ and $7-2\sqrt{13}$. Hence, the families of solutions of (1.7) are given by:

$$x-y\sqrt{13}=\pm(7\pm2\sqrt{13})(18+5\sqrt{13})^{2w}, \quad w\in\mathbb{Z}$$

and taking $x=x_w$, $y=y_w$ we obtain

$$x_w-y_w=\pm(7\pm2\sqrt{13})(649\pm180\sqrt{13})^{w}, \quad w\in\mathbb{N} \qquad \square$$

Considering the general solutions (3.1), we prove the following proposition:

Proposition 3. *The sequence* $(x_w)_w\geq 0$ *and* $(y_w)_w$ *verify respectively the following recursive formulae:*

(i) $x_{w+2}=1298x_{w+1}-x_w$ *and*

(ii) $y_{w+2}=1298y_{w+1}-y_w$.

Proof. It is clear that, with relation (3.1) we have $x_0=7$, $x_1=1279$. Next, relation (3.1) can be expressed in the form:

$$x_{w+1}-y_{w+1}\sqrt{13}=(649+180\sqrt{13})(x_w-y_w\sqrt{13})$$

or $x_{w+2}-y_{w+2}\sqrt{13}=(649+180\sqrt{13})^2(x_w-y_w\sqrt{13})$

whence $x_{w+2}-y_{w+2}\sqrt{13}=(842401+233640\sqrt{13})(x_w-y_w\sqrt{13})$.

But $842401+233640\sqrt{13}=842402+233640\sqrt{13}-1$.

Therefore $x_{w+2}-y_{w+2}\sqrt{13}=(842402+233640\sqrt{13})(x_w-y_w\sqrt{13})-(x_w-y_w-\sqrt{13})$

whence $x_{w+2}-y_{w+2}\sqrt{13}=1298(x_{w+1}-y_{w+1}\sqrt{13})(x_w-y_w\sqrt{13})$,

so that

$$x_{w+2}=1298x_{w+1}-x_w;$$

$$y_{w+2}=1298y_{w+1}+y_w$$

Put $\alpha = 649 + 180\sqrt{13}$ and write $\beta = 649 - 180\sqrt{13}$ for the conjugate of α. Then, α and β have the same minimal polynomial $P_\alpha = P_\beta = x^2 - 1298x + 1$. Relation (3.1) shows that we can write:

$$x_w - y_w\sqrt{13} = \varepsilon(7 + 2\varepsilon\sqrt{13})\alpha^w, \quad \varepsilon \pm 1; \tag{3.2}$$

$$x_w - y_w\sqrt{13} = \varepsilon(7 + 2\varepsilon\sqrt{13})\beta^w, \quad \varepsilon \pm 1; \tag{3.3}$$

$$x_w - y_w\sqrt{13} = (7 + 2\sqrt{13})\beta^w; \tag{3.4}$$

$$x_w - y_w\sqrt{13} = (72\sqrt{13})\alpha^w. \tag{3.5}$$

Then, the sequences $(x_w)_{w \geq 0}$ and $(y_w)_{w \geq 0}$ of relations (3.2), (3.3), (3.4) and (3.5) also verify respectively the recursive formulae (i) and (ii) of Proposition 3. □

Therefore in view of Propositions 2 and 3, it suffices to consider only relation (3.2). In this case, let us pass first the following remark:

Remark 4. *As* $z_w = \frac{y_w}{2}$ *(cf. second relation of* (1.6)*), the positive value* $z_0 = 1$ *corresponds to* $w = 0$ *in* (3.2)*. Then, since* v *is odd:* $v = 2t + 1$*, this imposes* $t = 0$ *and* $v = 1$*.*

Therefore, in the ensuing of the two following sections, we may suppose that $t \geq 1$. Then, the goal of our work is to find w such that $z_w = \pm 13^t$. Then, we look at z_w modulo ρ for well chosen integers, that is to say $\rho = 7, 13, 53, 79$.

4 Case $\varepsilon = 1$

In this case, we have $z_0 = 1$, $z_1 = 1279$. We now use Proposition 3 to prove the following proposition:

Proposition 5. *Let $\varepsilon = 1$ and let $w = 2t+1$ with $t \geq 1$. If the sequence $(y_w)_{w\geq 0}$ given by* (3.2) *satisfies the condition (ii) of the preceding proposition, then we have the following congruences:*

$$\begin{aligned}
(i) &\quad z_w \equiv \pm 1 \pmod{7};\\
(ii) &\quad z_w \equiv 0 \pmod{13};\\
(iii) &\quad z_w \equiv 16 \pmod{53};\\
(iv) &\quad z_w \equiv 13 \pmod{79}.
\end{aligned}$$

Proof. Considering relation (3.2) for $\varepsilon = 1$, we must find w such that

$$x_w^2 - 13y_w^2 = -3. \tag{4.1}$$

Hence, with (ii) of Proposition 3:

(i) modulo 7 we have:

w	0	1	2	3	4	5	6	7
z_w	1	5	0	2	6	2	0	5

Here, we see that the sequence (z_w) is periodic with period 8. Then, $z_w = \pm 13^t$ implies $z_w \equiv 1 \pmod{7}$.

(ii) modulo 13 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
z_w	1	5	2	4	3	3	4	2	5	1	6	0	7	12	8	11	3	10	10

w	19	20	21	22	23	24	25
z_w	9	11	8	12	7	0	6

Here, we see that the sequence (z_w) is periodic with period 26. Then, $z_w = \pm 13^t$ implies (as we have assumed that $t \geq 1$) $z_w \equiv 0 \pmod{13}$.

(iii) modulo 53 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12
z_w	1	7	22	35	40	51	14	48	15	24	26	16	19.

We see that the sequence (z_w) is periodic with period 13. Then, $w \equiv 11 \pmod{13}$ shows that $z_w \equiv 16 \pmod{53}$.

(iv) modulo 79 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12
z_w	1	15	35	69	20	58	56	29	61	70	28	13	19.

We see that the sequence (z_w) is periodic with period 13. Then, $w \equiv 11 \pmod{13}$ shows that $z_w \equiv 13 \pmod{79}$. □

In the next theorem we go on to prove that (1.1) has no integer solution.

Theorem 6. *Let* $\varepsilon = 1$ *be and let* $w = 2t+1$ *with* $t \geq 1$. *Then, under the congruences of the precedent proposition, equation* (1.1) *has no solution in* $\mathbb{Z}$.

Proof. Since $\varepsilon = 1$, with the precedent proposition we have:
the congruence of (i) imposes $w \equiv 0, 4 \pmod{8}$, therefore 4 divides w and this is impossible.
The congruence of (ii) also imposes $w \equiv 11, 24 \pmod{26}$; but w is even, therefore $w \equiv 24 \pmod{26}$, that is to say $w \equiv 12 \pmod{13}$: impossible.
From (iii), we see that 13 is of order 13 and we have:

t	1	2	3	4	5	6	7	8	9	10	11	12	13
13^t	13	10	24	47	28	46	15	36	44	42	16	49	1

and $z_w = 13^t$ implies $13^t \equiv 16$ whence $t \equiv 11 \pmod{13}$: impossible because $t \equiv 1 \pmod{13}$;
$z_w = 13^t$ implies $13^t \equiv 37$: impossible.
From (iv) we have: $z_w = \pm 13^t$ implies $13^{t-1} \equiv 1 \pmod{79}$ and as 13 is of order 39

modulo 79, we must have:
$13^{t-1} \equiv 1 \pmod{79}$, impossible because 13 is of the odd order modulo 79;
$13^t - 1 \equiv 1 \pmod{79}$, $t \equiv 1 \pmod{79}$ and $t \equiv 1 \pmod{13}$. This proves that equation (1.1) has no solution when $\varepsilon = 1$ and $w = 2t+1$ with $t \geq 1$. □

5 Case $\varepsilon = -1$

In this case we have $z_0 = 1$, $z_1 = 19$ and as in the preceding case, we also use Proposition 3 to prove the following proposition:

Proposition 7. *Let* $\varepsilon = 1$ *be and let* $w = 2t+1$ *with* $t \geq 1$. *If the sequence* $(y_w)_{w\geq 0}$ *given by* (3.2) *satisfies the condition (ii) of Proposition 3, then we have the following congruences:*

(i) $z_w \equiv 1 \pmod{7}$*;*

(ii) $z_w \equiv 0 \pmod{13}$*;*

(iii) $z_w \equiv 16 \pmod{53}$*;*

(iv) $z_w \equiv 13 \pmod{79}$.

Proof. Considering relation (3.2) for $\varepsilon = 1$, we must find also w such that

$$x_w^2 - 13y_w^2 = 3. \tag{5.1}$$

Hence, with (ii) of Proposition 3 we have:
(i) modulo 7, the sequence (z_w) is periodic with period 8 and it is exactly the same as in the case $\varepsilon = 1$.
(ii) modulo 13 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
z_w	1	6	0	7	12	8	11	3	10	10	9	11	8	12	7	0	0	1	5

w	19	20	21	22	23	24	25
z_w	2	4	3	3	4	2	5.

Here, we see that the sequence (z_w) is periodic with period 26. Then, $z_w = \pm 13^t$ implies (as $t \geq 1$) $z_w \equiv 0 \pmod{13}$.

(iii) modulo 53 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12
z_w	1	19	16	26	24	15	48	14	51	40	35	22	7.

We see that the sequence (z_w) is periodic with period 13. Then, $w \equiv 2 \pmod{13}$ shows that $z_w \equiv 16 \pmod{53}$.

(iv) modulo 79 we have:

w	0	1	2	3	4	5	6	7	8	9	10	11	12
z_w	1	19	13	28	70	61	29	56	58	20	69	35	15.

We see that the sequence (z_w) is periodic with period 13. Then, $w \equiv 2 \pmod{13}$ shows that $z_w \equiv 13 \pmod{79}$. $\square$

Next, we go on to prove the following theorem:

Theorem 8. *Let $\varepsilon = 1$ be and let $w = 2t+1$ with $t \geq 1$. Then, under the congruences of the precedent proposition, equation* (1.1) *has no solution in* $\mathbb{Z}$.

Proof. Since $\varepsilon = 1$, with the precedent proposition we have:

the congruence (i) imposes $w \equiv 0, 4 \pmod{8}$, therefore 4 divides w and this is impossible.

The congruence (ii) imposes $w \equiv 2, 15 \pmod{26}$; but w is even: $w \equiv 2 \pmod{26}$: therefore $w \equiv 2 \pmod{13}$: impossible.

The proof of (iii) is the same as in Theorem 6. From (iv) we have: $w \equiv 2 \pmod{13}$ implies $z_w \equiv 13 \pmod{79}$ and we conclude as in the case $\varepsilon = 1$.

This proves that equation (1.1) has no solution when $\varepsilon = 1$ and $w = 2t+1$ with $t \geq 1$. $\square$

6 Complete set of solutions of (1.1)

This section is devoted to the theorem giving all the solutions of (1.1). For the proof of this theorem we shall use the parity of the positive integer v (cf. Section 1), Theorem 1 and Remark 4.

Theorem 9. *Equation* (1.1) *has only:*
(i) the trivial solutions $(v,w) = (0,-1),\ (0,0)$*, when* $v \equiv 0 \pmod 2$*;*
(ii) the non trivial solutions $(v,w) = (1,-4),\ (1,3)$*, when* $v \equiv 1 \pmod 2$.

Proof. (i) When $v \equiv 0 \pmod 2$, from Theorem 1 we have (as $z = 13^t$):
if $z = 0$, this implies $13^t = 0$: impossible;
if $z = -1$, this implies $13^t = -1$: impossible;
if $z = 1$, then we have $13^t = 1$: this imposes $t = 0$, whence $v = 0$; then equation (1.3) becomes $w^2 + w = 0$ which has the solutions $w = 0$ and $w = -1$.
Finally, when v is even, (1.1) has only the trivial solutions $(v,w) = (0,-1),\ (0,0)$.
(ii) When $v \equiv 1 \pmod 2$, from Remark 4, we have $v = 1$ and we see that the solutions of equation $13 - w^2 - w = 1$ are $w = 3$ and $w = -4$. Moreover, Theorem 6 and 8 show that there is no more solution. Finally when v is odd, (1.1) has only the solutions $(v,w) = (1,4),\ (1,3)$. $\square$

Remark 10. *Taking* $a = b = c = d = 1$ *and* $y = 13$ *in* (1.2)*, we obtain the equation* $x^2 + x + 1 = 13^n$*: this is the same as equation* (1.1)*. But Theorem 9 shows that in certain cases, the integer n satisfying* (1.2) *is not forcely 3.*

Therefore, equation (1.1) *is a special case of equation* (1.2).

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